1 Introduction

This document serves as a reference for the mathematical framework used to represent the polarization of electromagnetic waves and the polarimetric response of instrumentation.

2 General Equations

2.1 Polarization

A monochromatic wave traveling in the positive z direction is given by

$$e(t) = a \exp i(\omega t - \kappa z). \tag{1}$$

At $z = 0$, a quasi-monochromatic electromagnetic wave is represented by

$$\boldsymbol{e}(t) = \begin{pmatrix} e_0(t) = a_0(t) \exp i(\omega t + \phi_0(t)) \\ e_1(t) = a_1(t) \exp i(\omega t + \phi_1(t)) \end{pmatrix}. \tag{2}$$

The coherency matrix is defined as

$$\boldsymbol{\rho} = \langle \boldsymbol{e}(t) \otimes \boldsymbol{e}^\dagger(t) \rangle = \begin{pmatrix} \langle e_0(t) e_0^*(t) \rangle & \langle e_0(t) e_1^*(t) \rangle \\ \langle e_1(t) e_0^*(t) \rangle & \langle e_1(t) e_1^*(t) \rangle \end{pmatrix}, \tag{3}$$

where $\boldsymbol{e}^\dagger$ is the Hermitian transpose of $\boldsymbol{e}$ and the angular brackets denote an ensemble average. The coherency matrix may be written as a linear combination of Hermitian basis matrices

$$\boldsymbol{\rho} = \frac{1}{2} \sum_{k=0}^{3} S_k \boldsymbol{\sigma}_k = (S_0\, \boldsymbol{\sigma}_0 + \boldsymbol{S} \cdot \boldsymbol{\sigma})/2, \tag{4}$$

where $\boldsymbol{\sigma}_0$ is the 2×2 identity matrix, $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3)$ are the Pauli spin matrices, S_0 is the total intensity, or Stokes I, and $\boldsymbol{S} = (S_1, S_2, S_3)$ is the Stokes polarization vector. The Stokes parameters may also be expressed in terms of the coherency matrix

$$S_k = \mathrm{tr}(\boldsymbol{\sigma}_k \boldsymbol{\rho}), \tag{5}$$

where tr() is the matrix trace operator. The Pauli matrices are

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{6}$$

2.1.1 Geometric Interpretation

The three-dimensional space of the Stokes polarization vector, $\boldsymbol{S}$, is spanned by the orthonormal basis vectors, $\hat{\boldsymbol{s}}_{\boldsymbol{k}}$, such that $\hat{\boldsymbol{s}}_{\boldsymbol{k}} \cdot \boldsymbol{S} = S_k$ and $\hat{\boldsymbol{s}}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma}_k$. Stokes Q, U, and V are calculated by the projection of $\boldsymbol{S}$ onto the Stokes unit vectors, $\hat{\boldsymbol{q}}$, $\hat{\boldsymbol{u}}$, and $\hat{\boldsymbol{v}}$, respectively. This relationship is summarized by $\boldsymbol{p} = (Q, U, V) = \mathbf{R}^T \boldsymbol{S}$, where $\mathbf{R} = (\hat{\boldsymbol{q}}\ \hat{\boldsymbol{u}}\ \hat{\boldsymbol{v}})$ is a three-dimensional rotation matrix with columns defined by the Stokes unit vectors. The orientation of these basis vectors with respect to $\hat{\boldsymbol{s}}_{\boldsymbol{k}}$ depends upon the reference frame in which the electric field vector is represented.

2.2 Polarimetric Transformations

Linear transformations of the electric field are represented by complex 2×2 Jones matrices. Under the operation, $\boldsymbol{e}'(t) = \mathbf{J}\boldsymbol{e}(t)$, the coherency matrix is subjected to a congruence transformation, $\boldsymbol{\rho}' = \mathbf{J}\boldsymbol{\rho}\mathbf{J}^\dagger$.

An arbitrary Jones matrix with unit determinant may be parameterized by its polar decomposition, $\exp[(\beta\,\hat{\boldsymbol{m}} + i\phi\,\hat{\boldsymbol{n}})\cdot\boldsymbol{\sigma}]$. The Hermitian matrices,

$$\mathbf{B}_{\hat{\boldsymbol{m}}}(\beta) = \exp(\beta\,\hat{\boldsymbol{m}} \cdot \boldsymbol{\sigma}) = \cosh\beta\,\boldsymbol{\sigma}_0 + \sinh\beta\,\hat{\boldsymbol{m}} \cdot \boldsymbol{\sigma}, \tag{7}$$

effect a Lorentz boost of the Stokes 4-vector along the axis $\hat{\boldsymbol{m}}$ by an impact parameter 2β. Likewise, the unitary matrices,

$$\mathbf{R}_{\hat{\boldsymbol{n}}}(\phi) = \exp(i\phi\,\hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}) = \cos\phi\,\boldsymbol{\sigma}_0 + i\sin\phi\,\hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}, \tag{8}$$

rotate the Stokes polarization vector about the axis $\hat{\boldsymbol{n}}$ by an angle 2ϕ.

2.3 Cartesian (Linear) Basis

In the Cartesian (or linear) basis, the plane wave propagates toward the observer along the z-axis, and $e_0 = e_x$ and $e_1 = e_y$ are the components of the electric field projected onto North and East, respectively; the Stokes vector, $\boldsymbol{S} = (Q, U, V)$, and

$$\boldsymbol{\rho} = \frac{1}{2}\begin{pmatrix} I + Q & U - iV \\ U + iV & I - Q \end{pmatrix}. \tag{9}$$

By comparison of equations 3 and 9

$$U + iV = 2\langle e_x^* e_y \rangle = 2\langle a_x a_y \exp i(\phi_y - \phi_x) \rangle. \tag{10}$$

That is, Stokes V is positive when the phase of e_y **leads** that of e_x. According to Hamaker & Bregman (1996 A&ASS 117:161), the IEEE definition of circular polarization is such that:

> For right-handed circular polarization, the position angle of the electric vector at any point increases with time; this implies that the y component of the field lags the x component.

Therefore, in our reference frame, Stokes V is positive for left-handed circular polarization (LCP). Although this is opposite to the IAU convention, in which Stokes V is positive for right-handed circular polarization (RCP), it is consistent with other authors, including Kraus and Born & Wolf.

2.4 Circular Basis

In the circular basis, $e_0 = e_l = \boldsymbol{e}_l^\dagger \boldsymbol{e}$, and $e_1 = e_r = \boldsymbol{e}_r^\dagger \boldsymbol{e}$, where $\boldsymbol{e}$ is the electric field vector in the linear basis,

$$\boldsymbol{e}_l = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad \boldsymbol{e}_r = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}. \tag{11}$$

The transformation from a linear to circular basis is then

$$\mathbf{C} = (\boldsymbol{e}_l \; \boldsymbol{e}_r)^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}. \tag{12}$$

Under the congruence transformation, $\boldsymbol{\rho}' = \mathbf{C}\boldsymbol{\rho}\mathbf{C}^\dagger$, $\boldsymbol{S}' = (V, Q, U)$. The basis vectors $\boldsymbol{e}_l$ and $\boldsymbol{e}_r$ are not the only possible representation of circularly polarized receptors. For example, it could be argued that

$$\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

is a better choice for the definition of $\mathbf{C}$, because then the same input Stokes parameters (1,0,1,0) will produce an equal and in-phase response in each receptor regardless of linear or circular basis. However, after a transformation with this matrix, $\boldsymbol{S}' = (V, U, -Q)$. The convention chosen in equation 12 is the only orthonormal basis (ignoring absolute phase) that effects the required cyclic permutation of the Stokes parameters. Its acceptance implies that Stokes Q produces equal, in-phase responses in each receptor.

3 Universal Feed Description

Under the congruence transformation, a Jones matrix has seven degrees of freedom. One common parameterization is described in the following section.

3.1 Phenomenological Model

The response of an ideal feed with two orthogonally polarized receptors is given by $\mathbf{S}(\theta, \epsilon) = \mathbf{R}_{\hat{\boldsymbol{u}}}(\epsilon)\mathbf{R}_{\hat{\boldsymbol{v}}}(\theta)$. Here, the receptors have ellipticity angles equal to ϵ and mutually perpendicular orientations defined by θ. Using this notation, a feed with non-orthogonal receptors is represented by

$$\mathbf{F} = \boldsymbol{\delta}_0\mathbf{S}(\theta_0, \epsilon_0) + \boldsymbol{\delta}_1\mathbf{S}(\theta_1, \epsilon_1), \tag{13}$$

where $\boldsymbol{\delta}_a$ is the 2×2 selection matrix, such that the product, $\boldsymbol{\delta}_a\mathbf{B}$, returns a matrix that contains only the a^{th} row of $\mathbf{B}$. The differential gain and phase of the instrument are represented by $\mathbf{B}_{\hat{\boldsymbol{s}}_1}(\gamma)$ and $\mathbf{R}_{\hat{\boldsymbol{s}}_1}(\varphi)$. Including the absolute gain, G, the Jones matrix that describes the instrumental response is

$$\mathbf{J} = G\,\mathbf{B}_{\hat{\boldsymbol{s}}_1}(\gamma)\mathbf{R}_{\hat{\boldsymbol{s}}_1}(\varphi)\mathbf{F}. \tag{14}$$

The seven scalar parameters are the gain G, the differential gain γ, the differential phase φ, the receptor orientations θ_{0-1}, and ellipticity angles ϵ_{0-1}.

3.2 Parameterization of Feed Configuration

Although it is possible to completely determine all seven degrees of freedom in the instrumental response, it is often not feasible. In most cases, a reference source is used to calibrate only the differential gain and phase of the backend. Even when it is possible to determine the receptor orientation and ellipticity angles, it may still be preferable to model them as small corrections to a known feed configuration. The following sections describe various ways to parameterize the known configuration of the feed and reference source.

3.2.1 Universal

As already stated, an ideal feed with two orthogonally polarized receptors is described by its orientation and ellipticity angles, θ and ϵ. For a linear feed, $\theta = \epsilon = 0$, and for a circular feed, $\theta = \epsilon = \pi/4$. The polarization of the reference source can also be completely described by an orientation and ellipticity angle pair.

3.2.2 Practical

Although the receptor angles are sufficient to completely describe an ideal feed, it may be preferable to use more commonly encountered and/or more readily measured quantities. The description may also be simplified by assuming that the reference source is linearly polarized. At Parkes, the following parameters are available:

- **basis**: circular or linear
- **hand**: left or right-handed
- **reference angle**: feed angle of the electric field vector that induces an equal, in-phase response in each receptor
- **calibrator phase**: differential phase ($\tan^{-1}(S_3/S_2)$) of the internal reference source

Linear Basis: The *reference angle* has a nominal value of 45 degrees. Since $S_2 = U$ and $S_3 = V = 0$, there are only two possible values for the *calibrator phase*: 0 ($U > 0$) and 180 ($U < 0$) degrees.

Circular Basis: The *reference angle* has a nominal value of 0 degrees. Since $S_2 = Q$ and $S_3 = U$, the *calibrator phase* is arbitrary.

Given these parameters, the Jones matrix of the feed is the product

$$\mathbf{J} = \mathbf{XCR}_{\hat{\boldsymbol{v}}}(\Theta), \tag{15}$$

where $\mathbf{C}$ is the identity in the linear basis and equal to equation 12 in the circular basis, $\mathbf{X}$ is the identity in a right-handed system and the exchange matrix in a left-handed system, and Θ is equal to the *reference angle* minus its nominal value. The coherency matrix of the reference source is given by

$$\boldsymbol{\rho_c} = \frac{1}{2}\begin{pmatrix} 1 & \exp -i\Phi_c \\ \exp i\Phi_c & 1 \end{pmatrix}.$$

where Φ_c is the *calibrator phase*.

3.3 Complex Conjugation

Complex conjugation of the electric field can occur, for example, during lower sideband down conversion or when the design of an instrument is based upon a different convention for the sign of the phase than in equation 1. Complex conjugation cannot be represented by a product with a Jones matrix; it results in a sign change in S_3, or reflection through the $\hat{\boldsymbol{s}}_1$-$\hat{\boldsymbol{s}}_2$ plane.

4 Implementation

Each of the various properties described in the preceding sections have a corresponding parameter in the PSRFITS definition and a representation in the PSRCHIVE software.

4.1 PSRFITS

The following table summarizes the properties that are described by the PSRFITS file format, the range of acceptable values for each parameter, and the effect that they have on calibrated data.

Table 1: PSRFITS header parameters related to polarimetric corrections.

Parameter	PSRFITS Name	Range	Effect	
			Linear	Circular
Backend Phase	BE_PHASE	+/-1	+/- V	+/- U
Downconversion[a]	BE_DCC	0/1	+/- V	+/- U
Feed Basis	FD_POLN	LIN or CIRC	(Q,U,V)	(V,Q,U)
Feed Hand	FD_HAND	+/- 1	+/- Q&V	+/- U&V
Reference Angle	FD_SANG	$-\pi/2 < \theta < \pi/2$	$\mathbf{R}_{\hat{\boldsymbol{v}}}(\Theta)$	
Calibrator Phase[b]	FD_XYPH	$-\pi < \Phi_c < \pi$	+/- U&V	$\mathbf{R}_{\hat{\boldsymbol{v}}}(\Phi_c/2)$

a Applies only when `Archive::bandwidth` is negative
b In the linear basis, $\Phi_c = 0$ or π

4.2 PSRCHIVE

Polarization parameters are stored in the `Backend` and `Receiver` classes, both of which inherit the `Archive::Extension` base class.

4.2.1 Backend Phase

The backend phase is stored in the `argument` attribute of the `Backend` class. The correction of phase conjugation is performed in the `correct_backend` method of the `PolnCalibrator` class. Both source and calibrator observations are corrected.

4.2.2 Downconversion Conjugation Corrected

This flag is stored in the `downconversion_corrected` attribute of the `Backend` class. When this flag is false and the `bandwidth` attribute of the `Archive` class is negative, the correction of phase conjugation is performed in the `correct_backend` method of the `PolnCalibrator` class. Both source and calibrator observations are corrected.

4.2.3 Feed Basis

The feed basis is stored in the `basis` attribute of the `Receiver` class. This attribute controls the conversion between Stokes parameters and the coherency matrix, as performed by the `convert_state` method of the `PolnProfile` class, and by the `convert` and `coherency` functions defined in `Pauli.h`. The conversion functions in `Pauli.h` use the `Pauli::basis` attribute. Both source and calibrator observations are affected by a change in basis.

4.2.4 Feed Hand

The feed hand is stored in the `hand` attribute of the `Receiver` class. It impacts upon the Jones matrix returned by `Receiver::get_transformation`, which is used by the `calibrate` method of the `CorrectionsCalibrator` class. Only source observations are affected by the feed hand.

4.2.5 Reference Angle

The reference angle is stored in the `field_orientation` attribute of the `Receiver` class. This attribute impacts upon the Jones matrix returned by

Receiver::get_transformation, which is used by the calibrate method of the CorrectionsCalibrator class. Only source observations are affected by the reference angle.

4.2.6 Calibrator phase

The calibrator phase is stored in the reference_source_phase attribute of the Receiver class. This attribute determines the Stokes parameters returned by the Receiver::get_reference_source method, which is used by the children of the ReferenceCalibrator class to determine differential gain and phase.